

# Independence of Four Projective Criteria for the Weak Invariance Principle

Olivier Durieu\*

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## Abstract

Let  $(X_i)_{i \in \mathbb{Z}}$  be a regular stationary process for a given filtration. The weak invariance principle holds under the condition  $\sum_{i \in \mathbb{Z}} \|P_0(X_i)\|_2 < \infty$  (see [11], [2], [3]). In this paper, we show that this criterion is independent of other known criteria: the martingale-coboundary decomposition of Gordin (see [7], [8]), the criterion of Dedecker and Rio (see [4]) and the condition of Maxwell and Woodroffe (see [13], [14], [16], [17]).

Keywords: Weak invariance principle; Martingale approximation; Projective criterion.

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## 1 Introduction and Results

The aim of this paper is to study the relation between several criteria to get the weak invariance principle of Donsker in dependent case. In the paper [5], the independence between three of them is already shown. These criteria are the martingale-coboundary decomposition, the projective criterion of Dedecker and Rio and the Maxwell-Woodroffe condition. Here, we consider a fourth one ( $\sum_{i \in \mathbb{Z}} \|P_0(X_i)\|_2 < \infty$ ) and we show that it is independent of the three others. Let us begin by the statements of the four criteria.

Let  $(\Omega, \mathcal{A}, \mu)$  be a probability space and let  $T$  be a bijective bimeasurable transformation of  $\Omega$  preserving  $\mu$ . We assume  $(\Omega, \mathcal{A}, \mu, T)$  is an ergodic dynamical system. Let  $f : \Omega \rightarrow \mathbb{R}$  be a measurable function with zero mean. We recall that the process  $(f \circ T^i)_{i \in \mathbb{N}}$  satisfies the weak invariance principle if the process

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor tn \rfloor - 1} f \circ T^k, t \in [0, 1]$$

converges in distribution to a Gaussian process in the space  $D([0, 1])$  provided with the Skorohod topology (see Billingsley [1]). Let  $\mathcal{F}$  be a sub- $\sigma$ -algebra of  $\mathcal{A}$  such that  $T^{-1}\mathcal{F} \subset \mathcal{F}$ .

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\*E-mail: olivier.durieu@etu.univ-rouen.fr.

web page: <http://www.univ-rouen.fr/LMRS/Persopage/Durieu/>

Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS-Université de Rouen.

We denote by  $\mathcal{F}_i$  the  $\sigma$ -algebra  $T^{-i}\mathcal{F}$ . The function  $f$  is called regular with respect to the filtration  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$  if

$$\mathbb{E}(f|\mathcal{F}_{-\infty}) = 0 \text{ and } \mathbb{E}(f|\mathcal{F}_{+\infty}) = f.$$

In the sequel, we assume that  $f$  is a square integrable function and we write  $\mathbb{L}^p$  for  $\mathbb{L}^p(\mu)$ ,  $p \geq 1$ .

- The first criterion is the martingale-coboundary decomposition due to Gordin [7]. We will restrict our attention to the martingale-coboundary decomposition in  $\mathbb{L}^1$  (see [8], and [6] for a complete proof). We say that  $f$  admits such a decomposition if  $f = m + g - g \circ T$  where  $(m \circ T^i)_{i \in \mathbb{Z}} \subset \mathbb{L}^1$  is a martingale difference sequence and  $g \in \mathbb{L}^1$ . If  $m \in \mathbb{L}^2$ , then the central limit theorem holds. Further, if  $\frac{1}{\sqrt{n}} \max_{i \leq n} |g \circ T^i|$  goes to 0 in probability, the weak invariance principle holds (see Hall and Heyde [9]). If  $f$  is a regular function with respect to the filtration  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$  then the martingale-coboundary decomposition in  $\mathbb{L}^1$  is equivalent to

$$\sum_{i=0}^{\infty} \mathbb{E}(f \circ T^i | \mathcal{F}_0) \text{ and } \sum_{i=0}^{\infty} f \circ T^{-i} - \mathbb{E}(f \circ T^{-i} | \mathcal{F}_0) \text{ converge in } \mathbb{L}^1, \quad (1)$$

see Volný [15]. Remark that if the process  $(f \circ T^i)_{i \in \mathbb{Z}}$  is adapted to  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ , the second sum is equal to zero.

- The Dedecker and Rio criterion is satisfied if

$$\sum_{k=1}^{\infty} f \mathbb{E}(f \circ T^k | \mathcal{F}_0) \text{ converges in } \mathbb{L}^1. \quad (2)$$

According to Dedecker and Rio [4], in the adapted case, this condition implies the weak invariance principle.

- The Maxwell-Woodroffe condition (see [13]) is satisfied if

$$\sum_{n=1}^{\infty} \frac{\|\mathbb{E}(S_n(f) | \mathcal{F}_0)\|_2}{n^{\frac{3}{2}}} < +\infty \quad (3)$$

where  $S_n(f) = \sum_{i=0}^{n-1} f \circ T^i$ . In the adapted case, Peligrad and Utev [14] proved that this condition implies the weak invariance principle. In the general case, the weak invariance principle holds as soon as (3) and

$$\sum_{n=1}^{\infty} \frac{\|S_n(f) - \mathbb{E}(S_n(f) | \mathcal{F}_n)\|_2}{n^{\frac{3}{2}}} < +\infty,$$

(see Volný [16], [17]).

The independence between these three criteria is proved in [5]. Here we add a new criterion. Let us denote by  $H_k = \mathbb{L}^2(\mathcal{F}_k)$  the space of  $\mathcal{F}_k$ -measurable functions which are square integrable and denote by  $P_k$  the orthogonal projection operator onto the space  $H_k \ominus H_{k-1}$ . For  $f \in \mathbb{L}^2$ ,

$$P_k(f) = \mathbb{E}(f | \mathcal{F}_k) - \mathbb{E}(f | \mathcal{F}_{k-1}).$$

• Let  $f$  be a regular function for the filtration  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ . As a consequence of a result given by Heyde [12] (see [15] Theorem 6) the central limit theorem holds as soon as

$$\sum_{i \in \mathbb{Z}} \|P_0(f \circ T^i)\|_2 < \infty. \quad (4)$$

In the adapted case, this result and the weak invariance principle were proved by Hannan [10], [11] under the assumption that  $T$  is weakly mixing. Hannan's weak invariance principle was proved without the extra assumption by Dedecker and Merlevède [2], Corollary 3. Finally, in the general case, the weak invariance principle under (4) is due to Dedecker, Merlevède and Volný [3], Corollary 2.

Our main result is the following theorem.

**Theorem 1.** *Conditions (1), (2), (3) and (4) are pairwise independent: in all ergodic dynamical system with positive entropy, for each couple of conditions among the four, there exists an  $\mathbb{L}^2$ -function satisfying the first condition but not the second one.*

With the results of [5], it remains to prove the independence of (1), (2), (3) with (4).

## 2 Proof of Theorem 1

Let  $(\Omega, \mathcal{A}, \mu, T)$  be an ergodic dynamical system with entropy greater or equal than 1. Let  $\mathcal{B}$  and  $\mathcal{C}$  be two independent sub- $\sigma$ -algebra of  $\mathcal{A}$ . Let  $(e_i)_{i \in \mathbb{Z}}$  be a sequence of independent  $\mathcal{B}$ -measurable random variables in  $\{-1, 1\}$  such that  $\mu(e_i = -1) = \mu(e_i = 1) = \frac{1}{2}$  and  $e_i = e_0 \circ T^i$ ,  $i \in \mathbb{Z}$ . We denote by  $\mathcal{F}_0$  the  $\sigma$ -algebra generated by  $\mathcal{C}$  and  $e_i$  for  $i \leq 0$  and we set  $\mathcal{F}_i = T^{-i}\mathcal{F}_0$ . Note that the case of entropy in  $(0, 1)$  can be studied by using another Bernoulli shift.

We introduce three sequences with the following properties:

- $(\theta_k)_{k \in \mathbb{N}} \subset (0, +\infty)$ ;
- $(\rho_k)_{k \in \mathbb{N}} \subset (0, 1)$  such that  $\sum_{k \geq 0} \rho_k < 1$ ;
- $(N_k)_{k \in \mathbb{N}} \subset \mathbb{N}$  such that  $N_{k+1} > N_k$ .

We can always find a sequence  $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, 1)$  such that  $\sum_{k \geq 0} \theta_k N_k \sqrt{\varepsilon_k} < \infty$ . So, we fix such a sequence and denote by  $f$  the function defined by

$$f = \sum_{k=1}^{\infty} \theta_k e_{-N_k} \mathbb{1}_{A_k}$$

where the sequence of sets  $(A_k)_{k \in \mathbb{N}}$  verifies:

- $\forall k \in \mathbb{N}$ ,  $A_k \in \mathcal{C}$ ,
- the sets  $A_k$  are disjoint,
- $\exists a \in (0, 1)$ ,  $\forall k \in \mathbb{N}$ ,  $a\rho_k \leq \mu(A_k) \leq \rho_k$ ,
- $\forall k \in \mathbb{N}$ ,  $\forall i \in \{0, \dots, N_k\}$ ,  $\mu(T^{-i}A_k \Delta A_k) \leq \varepsilon_k$ .

The construction of these sets is done in detail in [5].

First, remark that the process  $(f \circ T^i)_{i \in \mathbb{Z}}$  is adapted to the filtration  $(\mathcal{F}_i)_{i \in \mathbb{Z}}$  (then  $f$  is regular) and

$$f \in \mathbb{L}^2 \text{ if and only if } \sum_{k \geq 1} \theta_k^2 \rho_k < \infty.$$

The next proposition is proved in [5].

**Proposition 2.** *For the function  $f$  previously defined,*

- i. (1)  $\Leftrightarrow \sum_{k \geq 1} \theta_k \rho_k \sqrt{N_k} < \infty$ ;
- ii. (2)  $\Leftrightarrow \sum_{k \geq 1} \theta_k^2 \rho_k \sqrt{N_k} < \infty$ ;
- iii. (3)  $\Leftrightarrow \sum_{n \geq 1} n^{-\frac{3}{2}} \left( \sum_{k \geq 1} \theta_k^2 \min(n, N_k) \rho_k \right)^{\frac{1}{2}} < \infty$ .

We can state an analogous result for condition (4).

**Proposition 3.** *For the function  $f$  previously defined,*

$$\sum_{i \in \mathbb{Z}} \|P_0(f \circ T^i)\|_2 < \infty \text{ if and only if } \sum_{k \geq 1} \theta_k \sqrt{\rho_k} < \infty.$$

This proposition is proved in Section 3.

## Counterexamples

Now we give two counterexamples proving Theorem 1.

1. We consider the function  $f$  defined by the sequences  $\theta_k = \frac{2^k}{k}$ ,  $\rho_k = \frac{1}{4^k}$ ,  $N_k = k$ . Then  $f \in \mathbb{L}^2$  and using Proposition 2, we get:
  - a)  $\sum_{k \geq 1} \theta_k \rho_k \sqrt{N_k} = \sum_{k \geq 1} \frac{1}{2^k \sqrt{k}} < \infty$  and then (1) is verified.
  - b)  $\sum_{k \geq 1} \theta_k^2 \rho_k \sqrt{N_k} = \sum_{k \geq 1} k^{-\frac{3}{2}} < \infty$  and then (2) is verified.
  - c)  $\sum_{k \geq 1} \theta_k^2 \min(n, N_k) \rho_k = \sum_{k \geq 1} \frac{\min(n, k)}{k^2}$ . But  $\sum_{k=1}^n \frac{1}{k} \leq 1 + \ln(n)$  and  $\sum_{k=n+1}^{\infty} \frac{n}{k^2} \leq 1$ . Then  $\sum_{n \geq 1} n^{-\frac{3}{2}} \left( \sum_{k \geq 1} \theta_k^2 \min(n, N_k) \rho_k \right)^{\frac{1}{2}} \leq \sum_{n \geq 1} n^{-\frac{3}{2}} \sqrt{\ln(n) + 2} < \infty$  and (3) is verified.
  - d)  $\sum_{k \geq 1} \theta_k \sqrt{\rho_k} = \sum_{k \geq 1} \frac{1}{k}$  diverges and then by Proposition 3, (4) is not satisfied.

This counterexample shows that none of the conditions (1), (2) and (3) implies (4).

2. Now, if we consider the function  $f$  defined by the sequences:  $\theta_k = \frac{2^k}{k^{\frac{3}{2}}}$ ,  $\rho_k = \frac{1}{4^k}$ ,  $N_k = 2^{4k}$ , then  $f \in \mathbb{L}^2$  and:
  - a)  $\sum_{k \geq 1} \theta_k \rho_k \sqrt{N_k} = \sum_{k \geq 1} \frac{2^k}{k^{\frac{3}{2}}} \sqrt{2^{4k}} = \sum_{k \geq 1} \frac{2^k}{k^{\frac{3}{2}}} 2^{2k} = \sum_{k \geq 1} \frac{2^{5k}}{k^{\frac{3}{2}}}$  diverges and then (1) does not hold.
  - b)  $\sum_{k \geq 1} \theta_k^2 \rho_k \sqrt{N_k} = \sum_{k \geq 1} \frac{2^{2k}}{k^3} \sqrt{2^{4k}} = \sum_{k \geq 1} \frac{2^{6k}}{k^3}$  diverges and then (2) does not hold.

- c)  $\sum_{k \geq 1} \theta_k^2 \min(n, N_k) \rho_k = \sum_{k \geq 1} \frac{\min(n, 2^{4k})}{k^3} \geq n \sum_{k \geq \lfloor \frac{\ln n}{4 \ln 2} \rfloor} \frac{1}{k^3} \geq 8 \ln^2 2 \frac{n}{\ln^2 n}$ . Then  $\sum_{n \geq 1} n^{-\frac{3}{2}} \left( \sum_{k \geq 1} \theta_k^2 \min(n, N_k) \rho_k \right)^{\frac{1}{2}} \geq 8 \ln^2 2 \sum_{n \geq 1} \frac{1}{n \ln n}$  diverges and (3) does not hold.
- d) On the other hand,  $\sum_{k \geq 1} \theta_k \sqrt{\rho_k} = \sum_{k \geq 1} \frac{1}{k^{\frac{3}{2}}} < \infty$  and then (4) holds.

This shows that (4) does not imply any conditions (1), (2) or (3).

### Remarks

- i. In fact, we showed a little more than Theorem 1. We got that the three conditions (1), (2) and (3) together are independent of (4).
- ii. To show that (4) does not imply (1), it is enough to consider a linear process  $f = \sum_{i \in \mathbb{Z}} a_i \xi_i$  where  $(\xi_i)_{i \in \mathbb{Z}}$  is an iid sequence with  $\mu(\xi_0 = 1) = \mu(\xi_0 = -1) = \frac{1}{2}$  and  $a_i = \frac{1}{i^{\frac{3}{2}}}$ .

## 3 Proof of Proposition 3

First of all,  $(f \circ T^i)_{i \in \mathbb{Z}}$  is adapted to the filtration and then for all  $i < 0$ ,

$$P_0(f \circ T^i) = 0.$$

For  $i \geq 0$ , we have

$$\begin{aligned} P_0(f \circ T^i) &= \mathbb{E}(f \circ T^i | \mathcal{F}_0) - \mathbb{E}(f \circ T^i | \mathcal{F}_{-1}) \\ &= \sum_{k \geq 1} \theta_k [\mathbb{E}(e_{-N_k+i} | \mathcal{F}_0) - \mathbb{E}(e_{-N_k+i} | \mathcal{F}_{-1})] \mathbb{1}_{A_k} \circ T^i. \end{aligned}$$

Since  $e_j$  is  $\mathcal{F}_0$ -measurable for  $j \leq 0$  and independent of  $\mathcal{F}_0$  for  $j > 0$ ,

$$\begin{aligned} \mathbb{E}(e_{-N_k+i} | \mathcal{F}_0) - \mathbb{E}(e_{-N_k+i} | \mathcal{F}_{-1}) &= \begin{cases} e_0 & \text{if } i = N_k \\ 0 & \text{otherwise} \end{cases} \\ &= e_0 \mathbb{1}_{\{i=N_k\}}. \end{aligned}$$

Thus

$$\begin{aligned} P_0(f \circ T^i) &= \sum_{k \geq 1} \theta_k e_0 \mathbb{1}_{\{i=N_k\}} \mathbb{1}_{A_k} + \sum_{k \geq 1} \theta_k e_0 \mathbb{1}_{\{i=N_k\}} (\mathbb{1}_{T^{-i}A_k} - \mathbb{1}_{A_k}) \\ &= I_1(i) + I_2(i). \end{aligned}$$

For  $I_2$ , we use the fact that  $\mu(A_k \Delta T^{-i}A_k) \leq \varepsilon_k$  for  $0 \leq i \leq N_k$  to get

$$\begin{aligned} \|I_2(i)\|_2 &\leq \sum_{k \geq 1} \theta_k \mathbb{1}_{\{i=N_k\}} \|e_0 \mathbb{1}_{A_k \Delta T^{-i}A_k}\|_2 \\ &\leq \sum_{k \geq 1} \theta_k \mathbb{1}_{\{i=N_k\}} \sqrt{\varepsilon_k}. \end{aligned}$$

Remark for each  $i \geq 0$ , there is at most one integer  $k$  such that  $N_k = i$  and for each  $k \geq 1$ , there exists an integer  $i$  such that  $i = N_k$ . We deduce

$$\begin{aligned} \sum_{i \geq 0} \|I_2(i)\|_2 &\leq \sum_{i \geq 0} \sum_{k \geq 1} \theta_k \mathbb{1}_{\{i=N_k\}} \sqrt{\varepsilon_k} \\ &= \sum_{k \geq 1} \theta_k \sqrt{\varepsilon_k} \end{aligned}$$

which is finite by the assumptions.

Thus,  $\sum_{i \geq 0} \|P_i(f)\|_2$  is converging if and only if  $\sum_{i \geq 0} \|I_1(i)\|_2$  is converging. Now for a fixed  $i$ , since the sets  $A_k$  are disjoint and since there is at most one  $k$  such that  $N_k = i$ , we have

$$\begin{aligned} \|I_1(i)\|_2 &= \sqrt{\sum_{k \geq 1} \theta_k^2 \mathbb{1}_{\{i=N_k\}} \mu(A_k)} \\ &= \sum_{k \geq 1} \theta_k \mathbb{1}_{\{i=N_k\}} \sqrt{\mu(A_k)}. \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{i \geq 0} \|I_1(i)\|_2 &= \sum_{i \geq 0} \sum_{k \geq 1} \theta_k \mathbb{1}_{\{i=N_k\}} \sqrt{\mu(A_k)} \\ &= \sum_{k \geq 1} \theta_k \sqrt{\mu(A_k)}. \end{aligned}$$

We can conclude the proof using  $a\rho_k \leq \mu(A_k) \leq \rho_k$ .

□

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## References

- [1] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons Inc., New York, 1968.
- [2] Jérôme Dedecker and Florence Merlevède. The conditional central limit theorem in Hilbert spaces. *Stochastic Process. Appl.*, 108(2):229–262, 2003.
- [3] Jérôme Dedecker, Florence Merlevède, and Dalibor Volný. On the weak invariance principle for non-adapted sequences under projective criteria. *J. Theoret. Probab.*, 20(4):971–1004, 2007.
- [4] Jérôme Dedecker and Emmanuel Rio. On the functional central limit theorem for stationary processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 36(1):1–34, 2000.
- [5] Olivier Durieu and Dalibor Volný. Comparison between criteria leading to the weak invariance principle, to appear. *Ann. Inst. H. Poincaré Probab. Statist.*
- [6] Carl-Gustav Esseen and Svante Janson. On moment conditions for normed sums of independent variables and martingale differences. *Stochastic Process. Appl.*, 19(1):173–182, 1985.

- [7] M. I. Gordin. The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR*, 188:739–741, 1969.
- [8] M. I. Gordin. Abstracts of communication, t.1: A-k. International conference on probability theory, Vilnius. 1973.
- [9] P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. Probability and Mathematical Statistics.
- [10] E. J. Hannan. Central limit theorems for time series regression. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 26:157–170, 1973.
- [11] E. J. Hannan. The central limit theorem for time series regression. *Stochastic Process. Appl.*, 9(3):281–289, 1979.
- [12] C. C. Heyde. On the central limit theorem for stationary processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 30:315–320, 1974.
- [13] Michael Maxwell and Michael Woodroffe. Central limit theorems for additive functionals of Markov chains. *Ann. Probab.*, 28(2):713–724, 2000.
- [14] Magda Peligrad and Sergey Utev. A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.*, 33(2):798–815, 2005.
- [15] Dalibor Volný. Approximating martingales and the central limit theorem for strictly stationary processes. *Stochastic Process. Appl.*, 44(1):41–74, 1993.
- [16] Dalibor Volný. Martingale approximation of non adapted stochastic processes with nonlinear growth of variance. In *Dependence in probability and statistics*, volume 187 of *Lecture Notes in Statist.*, pages 141–156. Springer, New York, 2006.
- [17] Dalibor Volný. A nonadapted version of the invariance principle of Peligrad and Utev. *C. R. Math. Acad. Sci. Paris*, 345(3):167–169, 2007.